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Low-lying spectra in anharmonic three-body oscillators with a strong short-range repulsion

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Abstract

One-dimensional three-body Schrödinger equation is studied with binding mediated by the power-law two-body long-range attraction $V^{(L)}(x) = \alpha^2 x^L$ in superposition with a short-range repulsion $V^{(-K)}(x) = \beta^2/x^K$ (plus, possibly, further subdominant power-law components). Such an unsolvable and non-separable generalization of Calogero model (where one had $L = K = 2$) is shown to become separable and solvable at all integers $K > 0$ and $L > 0$ in the limit of the large repulsion $\beta \gg 1$.

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1. Introduction

Solutions of an A -particle Schrödinger bound-state problem

$$\hat{H}\Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_A) = E\Psi(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_A) \quad (1)$$

are vital in atomic and nuclear physics as well as in quantum chemistry. Using some realistic input Hamiltonian \hat{H} one often calculates the low-lying spectrum of energies as a characteristic output. Its construction is not easy in general. Even the results of some very advanced (usually, variational) numerical methods need not be reliable and may require an independent verification, say, via, at least, a partially exactly solvable, simplified A -body model (their concise review appeared recently in [1]).

Of course, the most natural opportunity for the (partial) exact solvability occurs in the many-body Schrödinger equation (1) with the first non-trivial choice of $A = 3$. In the single dimension with $x_j \in \mathbb{R}$ and for the spinless and equal-mass particles, the solvability may play an independent important role in phenomenological as well as purely mathematical applications of the pertaining three-body models

$$H\Psi(x_1, x_2, x_3) = E\Psi(x_1, x_2, x_3) \quad H = -\sum_{i=1}^3 \frac{\partial^2}{\partial x_i^2} + \hat{W}(x_1, x_2, x_3) \quad (2)$$

especially when their binding is mediated by mere two-body forces,

$$\hat{W}(x_1, x_2, x_3) = V(x_1 - x_2) + V(x_2 - x_3) + V(x_3 - x_1). \quad (3)$$

In such a setting, in 1969 [2], Calogero discovered the exact solvability of the model which was based on the use of the most elementary spiked harmonic-oscillator (SHO) two-body forces

$$V^{(\text{SHO})}(x_i - x_j) = \omega^2(x_i - x_j)^2 + \frac{\nu(\nu + 1)}{(x_i - x_j)^2}. \quad (4)$$

This choice was motivated not only by the strong immediate phenomenological appeal of the SHO two-body potential itself but also by the encouraging fact that at the vanishing $\nu = 0$, the simplified Schrödinger equation (2)–(4) is well known to remain separable. This observation opened the path towards the closed $\nu \neq 0$ solution since in the polar Jacobi coordinates, its explicit construction emerges immediately after the reduction of the singular part of the sum (3) of potentials (4) to an elementary expression [2].

Our present paper may be perceived as a direct continuation of the same $A = 3$ project, paying attention just to a broader class of potentials characterized by their more general power-law behaviour in both the short- and long-range domains. These potentials may be chosen in the similar two-term form

$$V(x_i - x_j) = \alpha^2(x_i - x_j)^L + \beta^2(x_i - x_j)^{-K} \quad L, K = 1, 2, \dots \quad (5)$$

or in the more general polynomial form (A3) of appendix A. They may mimic many popular confining forces, say, of the Lennard-Jones type. They combine a suitable strongly repulsive singular core in the origin [3] with the standard anharmonic-oscillator asymptotics. One arrives, thus, at the most natural generalization of the Calogero forces, indeed.

We shall analyse the latter three-body problem in its two explicit partial differential forms (A4) and (A6) of appendix A. The main purpose of such a study was, originally, a search for all the possible remnants of the exact solvability which is, definitely, lost in its completeness at any $L > 2$ and/or $K > 2$. The project found initial encouragement also in our unexpected observation that the transition to $L = 4$ need not necessarily mean the loss of separability of equation (2) (see section 2 and, in particular, subsection 2.1). In parallel, we believed that even after a complete loss of separability, the problem might remain *approximately* separable and/or solvable.

In this direction, two main technical ingredients of our study of equation (2) consist in the use of the Jacobi coordinates at any K and L (cf appendix A) and in a successful generalization of the above-mentioned Calogero's trigonometric $K = 2$ identity to all the integer exponents K and L (cf appendix B). An ultimate incentive for the presentation of all these results came from our *a posteriori* observation that the latter identities prove to be amazingly elementary and compact.

In a way guided by a few examples revealing the simplifying role of the growth of β (and described in more detail in subsections 2.2 and 2.3) we are going to employ the simplicity of our auxiliary trigonometric identities of appendix B for conversion or re-arrangement of our partial differential (PD) Schrödinger equation (2) into a separable problem (i.e. into a set of two ordinary differential (OD) equations) plus corrections.

This idea in its application to our non-separable $A = 3$ problem is fully developed in section 3. We demonstrate there how the practical relevance of the separability-violating corrections decreases very quickly with increasing dominant repulsive component in superpositions (5). We show how our separable large- β approximants may be constructed as solvable and how they generate reliable low-lying three-body spectra. For an explicit

illustration of the efficiency of such a strategy, we first test it on the solvable $K = L = 2$ case (i.e. on the model of Calogero) in subsection 3.1.

In subsection 3.2 we move to the first ‘unsolvable’ example with $L = 3$ and show how the comparatively compact character of the trigonometric identities of appendix B enables us to achieve a full quantitative understanding of the competing mechanisms of repulsion and confinement in our non-separable two-dimensional effective potential wells with complicated shapes.

Our final illustrative example of subsection 3.3 employs $K = L = 3$ and represents, therefore, a full fledged modification of the Calogero model in both its short- and long-range interaction parts. Of course, this modification remains non-separable and is also exactly solvable only in the limit $\beta \rightarrow \infty$.

Section 4 is the summary of our results which re-emphasizes that all our three-body spiked anharmonic oscillators share the features demonstrated via particular examples. Thus, for all of them the low-lying spectra become available in the physically very appealing asymptotic domain characterized by very strong short-range repulsion in our two-body interactions.

2. Elementary guide: quartic oscillator

2.1. Quartic Schrödinger equation and its ‘forgotten’ separability

We need not recall the techniques presented in appendices A and B to see that the potential function (3) remains virtually trivial, in the Cartesian coordinates X and Y , for all the three-body regular harmonic oscillators (RHO) with $L = 2$ and $\beta = 0$ in (5). It is well known that the related PD Schrödinger equation (A4) with

$$U^{(2)}(X, Y) = 3X^2 + 3Y^2$$

degenerates to a pair of ordinary and solvable confluent hypergeometric differential equations. In polar coordinates, in addition, this two-dimensional RHO potential term becomes completely independent of the angular variable φ ,

$$\Omega^{(2)}(\varrho, \varphi) = U^{(2)}[X(\varrho, \varphi), Y(\varrho, \varphi)] = 3\varrho^2. \tag{6}$$

Still, to our great surprise we revealed that this example is not unique. There exists another two-body potential which leaves our PD Schrödinger equation (A6) separable. Such an assertion can be easily verified since in the context of appendix B.1 we may derive that

$$U^{(4)}(X, Y) = \frac{9}{2}(X^2 + Y^2)^2$$

i.e.

$$\Omega^{(4)}(\varrho, \varphi) = U^{(4)}[X(\varrho, \varphi), Y(\varrho, \varphi)] = \frac{9}{2}\varrho^4.$$

We see that the quartic oscillator belongs to a very exceptional set of three-body problems which are *rigorously* reducible to the solutions of an ordinary differential equation. Such an observation will definitely deserve a separate treatment, say, in the context of perturbation theory.

In what follows we are going to skip this opportunity because the separability of the quartic three-body oscillator seems so exceptional and unparalleled by any other power-law model. Indeed, in the light of the discussion in appendix B we may be certain that the spiked quartic anharmonic oscillator (SQAQO) two-body interaction

$$V^{(\text{SQAQO})}(x_i - x_j) = \omega^2(x_i - x_j)^2 + \lambda(x_i - x_j)^4 + \frac{\nu(\nu + 1)}{(x_i - x_j)^2} \tag{7}$$

generates the only set of PD Schrödinger equations (A6) which remain separable in the polar Jacobi coordinates.

2.2. The role of the pronounced two-body repulsion

In a marginal remark let us add that in contrast to the above-mentioned Calogerial $K = L = 2$ model, the separability of the latter quartic model does not imply its exact solvability. Although its two-body input force (7) is merely slightly more complicated than its Calogerial SHO predecessor (4), the SQAQO bound-state problem must be solved in terms of perturbation expansions or by a numerical, approximate method [4].

In spite of the (expected) loss of the exact solvability, the spiked quartic illustration may still benefit from the above-mentioned experience gained during the study of the exceptional force (4). One learns, in particular, that the apparently complicated expression

$$U^{(\text{SHO})}(X, Y) = 3\omega^2(X^2 + Y^2) + \frac{\nu(\nu + 1)}{2X^2} + \frac{\nu(\nu + 1)}{2(X - \sqrt{3}Y)^2} + \frac{\nu(\nu + 1)}{2(X + \sqrt{3}Y)^2}$$

emerging in the Calogero paper [2] may be significantly simplified by its further re-parametrization (cf equation (A5) in appendix A). As long as we now work with the same repulsive core, the same function occurs in the SQAQO case. We have to return to the Calogero ‘experimentally’ discovered trigonometric identity

$$\begin{aligned} & \frac{1}{2X^2} + \frac{1}{2(X - \sqrt{3}Y)^2} + \frac{1}{2(X + \sqrt{3}Y)^2} \\ & \equiv \frac{1}{2\varrho^2} \left(\frac{1}{\sin^2 \varphi} + \frac{1}{\sin^2(\varphi + \frac{2}{3}\pi)} + \frac{1}{\sin^2(\varphi - \frac{2}{3}\pi)} \right) = \frac{9}{2\varrho^2 \sin^2 3\varphi} \end{aligned} \quad (8)$$

which makes, obviously, the SQAQO version of the PD equation (A4) also manifestly separable.

Our fascination by the Calogero model concerns, first of all, the ease with which one may replace the difficult, PD Schrödinger equation (2) by a system of independent OD equations. In the same vein, the introduction of the spherical Jacobi coordinates replaces both the SHO and SQAQO versions of the PD problem (2) by the much more easily tractable ‘radial’ OD equation

$$\left[-\frac{d^2}{dr^2} + \frac{\nu(\nu + 1)}{r^2} + \omega^2 r^2 + \lambda r^4 - E \right] \psi(r) = 0 \quad r \in (0, \infty) \quad (9)$$

with $\lambda = 0$ or $\lambda > 0$, respectively.

It is well known that we have here $\nu = (A - 3)/2 + \ell$ where the integer index $\ell = 0, 1, \dots$ numbers the so-called (hyper)spherical harmonics [5]. One also recollects, vaguely, that the addition of a new force $V_{\text{spike}}(r) \sim g/r^2$ would remain tractable in equation (9) because its form coincides precisely with the kinetic (so-called centrifugal) singular term so that we just have to admit any real $\nu = \nu(g)$ [3].

For the time being, let us fix $\lambda = 0$ and pay attention just to the exactly solvable spiked harmonic oscillators with a strong repulsion $g \gg 1$ (i.e. with a large real $\nu \gg 1$). Of course, their spectrum $E_n = \omega(4n + 2\nu + 3)$ with $n = 0, 1, \dots$ is equidistant and merely moves upwards with the steady growth of ν . Even without any knowledge of the exact formulae, this phenomenon is easily explained by an elementary observation that near the minimum, the *shape* of the SHO potential term in equation (9) may be very well approximated by a one-dimensional RHO well. Thus, we may start from the formal Taylor series

$$\begin{aligned} V_{\text{eff}}(r) &= \frac{\nu(\nu + 1)}{r^2} + \omega^2 r^2 = V_{\text{eff}}(R) + \frac{1}{2}(r - R)^2 V''_{\text{eff}}(R) + \mathcal{O}[(r - R)^3 V'''_{\text{eff}}(R)] \\ &= 2\omega^2 R^2 + 4\omega^2(r - R)^2 + \mathcal{O}\left[\frac{(r - R)^3}{R}\right]. \end{aligned} \quad (10)$$

In the vicinity of the absolute minimum at $r = R = [\nu(\nu + 1)/\omega^2]^{1/4}$ and in the asymptotic domain characterized by the ‘thick’ spike term, $\nu \gg 1$, the latter potential is very well approximated by its first two terms since $1/R \approx 1/\sqrt{\nu} \ll 1$.

One arrives at a consistent picture where the approximate energy levels calculated for the one-dimensional leading-order RHO well, i.e. the values

$$\tilde{E}_m = V_{\text{eff}}(R) + 2\omega(2m + 1) \quad m = 0, 1, \dots$$

coincide with the above exact low-lying SHO spectrum E_n, E_1, \dots up to the second-order term in $1/R$. This is an extremely encouraging observation which shows that the domain where the short-range repulsion is large is, in a way, privileged.

One should keep in mind that in our exactly solvable illustration, the parameter $1/R$ is precisely the appropriate measure of smallness of corrections. Once its value proves sufficiently small, we arrive at an asymptotic equivalence of the two models,

$$\text{RHO} \longleftrightarrow \text{SHO}. \tag{11}$$

In addition, we may easily move beyond the leading-order precision, making use of the textbook perturbation series [6]. Their explicit construction to arbitrary order and for virtually any potential may proceed, e.g., along the lines described in full detail in the review paper [7].

The same observations may be easily extended to the ‘purely numerical’ SQAQ problem where the effective potential $V_{\text{eff}}(r) = \frac{\nu(\nu+1)}{r^2} + \omega^2 r^2 + \lambda r^4$ with $r = R + x$ may equally well be expanded in the similar Taylor series

$$\left[\frac{\nu(\nu + 1)}{R^2} + \omega^2 R^2 + \lambda R^4 \right] + 2 \left[\omega^2 R - \frac{\nu(\nu + 1)}{R^3} + 2\lambda R^3 \right] x + \dots$$

We see that the point R becomes an absolute minimum of $V_{\text{eff}}(r)$ whenever the linear, $\mathcal{O}(x)$ term vanishes. This specifies the value of R exactly (via a positive root of the cubic algebraic equation) and enables us to find the subsequent quadratic and higher Taylor-series terms

$$\dots + \left[\omega^2 + 3\frac{\nu(\nu + 1)}{R^4} + 6\lambda R^2 \right] x^2 + 4 \left[-\frac{\nu(\nu + 1)}{R^5} + \lambda R \right] x^3 + \mathcal{O}(x^4) \tag{12}$$

in closed form.

2.3. Another example possessing an inverse-cubic core

In the previous section we have chosen the exactly solvable illustration just to emphasize the key idea of construction which lies in the coincidence of the shapes of two different potentials near their minima, *caused* by the growing strength of the singularity in the origin. Let us now verify the applicability of such an approach using a slightly less trivial form of the single-particle repulsion,

$$\left[-\frac{d^2}{dr^2} + W(r) - E \right] \Psi(r) = 0 \quad W(r) = Fr^3 + \frac{G}{r^3} \quad r, F, G > 0. \tag{13}$$

At a large value of $G \gg 1$, potential $W(r)$ may be represented by its Taylor series again. Near its absolute minimum which occurs at $r_{\text{min}} = R = (G/F)^{1/6} \gg 1$ we have

$$W(R + \xi) = 2\sqrt{GF} + 9\sqrt{GF} \frac{\xi^2}{R^2} - 9\sqrt{GF} \frac{\xi^3}{R^3} + \dots \tag{14}$$

This expansion illustrates the feasibility of the same harmonic-oscillator approximation approach to our generic ‘unsolvable example’ (UE). In the light of the large- ν equivalence SHO \leftrightarrow RHO between the two exceptional but ‘equally exactly’ solvable examples, the

unsolvable equation (13) and its asymptotic solvability opens a way towards a triple asymptotic equivalence between the *three* low-lying spectra,

$$UE \longleftrightarrow RHO \longleftrightarrow SHO. \tag{15}$$

This means that we may, alternatively, also approximate any unsolvable model by the zero-order solvable model with a singularity. Of course, the triple-osculation property (15) restricts our choice of the spiked SHO model characterized, for clarity, by the zero subscript $_0$,

$$W_0(R_0 + \xi) = F_0(R_0 + \xi)^2 + \frac{G_0}{(R_0 + \xi)^2} = 2\sqrt{G_0 F_0} + 4F_0 \xi^2 - 4\sqrt{\frac{F_0^5}{G_0}} \xi^3 + \dots \tag{16}$$

We must require that the minima are the same, $R_0 = R$, and that the two harmonic oscillator parts of the wells (14) and (16) differ at most by a convenient constant shift. These requirements form a set of two equations with easy solution,

$$F_0 = \frac{9}{4} \sqrt[6]{GF^5} \quad G_0 = \frac{4}{9} \sqrt[6]{G^5 F}.$$

In the domain of large $G \gg 1$ we have $G_0 \gg F_0$ as expected.

We may summarize that all the unsolvable OD examples may be studied via *different* implementations of the *same* idea of the strong-repulsion approximation. Of course, even when all the available free parameters coincide as they should, one may still test and compare the *practical numerical* performance of the RHO and SHO alternatives. This will not be pursued here. We only shortly emphasize that the use of the more sophisticated two-parametric SHO zero-order forces with a core gives a better chance for an optimal choice of the zero-order approximant. *A priori*, an additional benefit might be spotted in the better coincidence of the domains (i.e. the half-axes of the coordinates) which might also support the preference of the pair of the UE and SHO Hamiltonian operators. Thus, the similarity of the shape (often called osculation) of the given curve $W^{(UE)}$ near its minimum with the solvable spiked well $W_0^{(SHO)}$ appears to be better founded, in spite of its slightly less immediate construction. Presumably, the importance of similar arguments might further increase at $A = 3$, in the genuine PD context.

3. Three-body spectra at any K and L

3.1. Solvable guide: the $K = L = 2$ model at $\nu \gg 1$

The only non-constant angular dependence in the Calogero $\Omega^{(SHO)}(\varrho, \varphi)$ occurs in its short-range repulsion component $\Omega^{(-2)}(\varrho, \varphi) \sim \sin^{-2} 3\varphi$. For $\varphi \in (0, \pi/3)$ (i.e. within a physical wedge in the phase space which is called, sometimes, the Weyl chamber), this function has a unique minimum at $\varphi_{\min} = \pi/6$ where $\sin 3\varphi_{\min} = 1$. In its vicinity, the angular Taylor series may be calculated as,

$$\frac{1}{\sin^2 3(\gamma + \pi/6)} = 1 + 9\gamma^2 + 54\gamma^4 + \frac{1377}{5}\gamma^6 + \dots$$

This means that the absolute minimum of the potential $\Omega^{(SHO)}(\varrho, \varphi)$ will lie at $\varphi = \varphi_{\min} = \pi/6$. For the determination of its second coordinate $\varrho = R$, we may analyse the function $\Omega^{(SHO)}(\varrho, \pi/6) = V_{\text{eff}}(\varrho)$ of the single variable $\varrho > 0$. The second necessary Taylor series is then very easily derived,

$$\begin{aligned} V_{\text{eff}}(\varrho) &= \frac{9\nu(\nu + 1)}{2\varrho^2} + 3\omega^2 \varrho^2 = V_{\text{eff}}(R) + \frac{1}{2}(\varrho - R)^2 V''_{\text{eff}}(R) + \mathcal{O}[(\varrho - R)^3 V'''_{\text{eff}}(R)] \\ &= 6\omega^2 R^2 + 12\omega^2(\varrho - R)^2 + \mathcal{O}\left[\frac{(\varrho - R)^3}{R}\right] \quad R = \sqrt[4]{\frac{3\nu(\nu + 1)}{2\omega^2}}. \end{aligned} \tag{17}$$

Obviously, we just have to assume that $R \gg 1$ in order to achieve full consistency of the Calogero model with its two-dimensional RHO *separable* approximation,

$$\Omega^{(\text{SHO})}(R + \xi, \eta/R + \pi/6) \approx 6\omega^2 R^2 + 12\omega^2 \xi^2 + 27\omega^2 \eta^2. \tag{18}$$

In a way predicted by the single-particle guide of section 2 one discovers that all the higher-order terms $\mathcal{O}(\eta^m \times \xi^n)$ remain suppressed by the same factor $1/R^{m+n-2} \ll 1$. Thus, in the domain of large ν , the low-lying spectrum of energies may be approximated by the RHO formula

$$E = E_{m,n} = 6\omega^2 R^2 + 2\sqrt{3}\omega(2n + 1) + 3\sqrt{3}\omega(2m + 1) + \dots \tag{19}$$

Up to the higher-order corrections, this prediction ‘explains’ the well-known exact spectrum of the low-lying energies in the Calogero model [2].

We may conclude that in spite of the use of polar coordinates, the model is suitable for the two-dimensional RHO approximation using the original Cartesian coordinates rotated merely by $\pi/6$. This is an immediate consequence of the structure of the kinetic term in equation (A6),

$$-\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\varrho^2} \frac{\partial^2}{\partial \varphi^2}.$$

Obviously, it may be *locally* re-interpreted as a Cartesian kinetic energy,

$$-\frac{\partial^2}{\partial \varrho^2} - \frac{1}{(R + \xi)^2} \frac{\partial^2}{\partial \varphi^2} = -\frac{\partial^2}{\partial \xi^2} - \left[1 + \mathcal{O}\left(\frac{\xi}{R}\right) \right] \frac{\partial^2}{\partial \eta^2}$$

with the re-scaling of one of the coordinates which we also employed in equation (18).

3.2. $A = 3$ confinement and its sample proof for the non-separable $L = 3$ oscillator

One of the simplest non-separable three-body models may be based on the cubic anharmonic-oscillator two-body potentials

$$V^{(\text{CAHO})}(x_i - x_j) = \omega^2(x_i - x_j)^2 + \gamma(x_i - x_j)^3 + \frac{\nu(\nu + 1)}{(x_i - x_j)^2}. \tag{20}$$

Of course, one must proceed with a certain care since even in the single-particle case, the cubic potential is not confining on the real line [8]. In the present setting, fortunately, we need not work on the whole real line. In contrast, due to the presence of the strong repulsion, we *must* parallel the Calogero considerations [2] and fix the ordering *or* numbering of our three particles in advance, keeping in mind that in the quantum case, our particles cannot tunnel through the $1/(x_i - x_j)^2$ barriers. This means that we may demand that, say, $x_1 > x_2 > x_3$. This automatically guarantees that our system will ‘live’ in a single ‘physical’ wedge with $\varphi \in (0, \pi/3)$. At this moment we may also guarantee that the cubic component $\Omega^{(3)}(\varrho, \varphi)$ of the whole potential $\Omega^{(\text{CAHO})}(\varrho, \varphi)$ will remain non-negative (i.e. confining or vanishing) at all the positive couplings $\gamma > 0$.

In a certain challenging extreme, let us now drop the safely confining quadratic force in equation (20) and show that even at $\omega = 0$, the three-body bound states will still remain localized. In essence, we have to demonstrate that the spiked cubic (SC) two-body forces $V^{(\text{SC})}(x_i - x_j)$ ($= V^{(\text{CAHO})}(x_i - x_j)$ at $\omega = 0$) will guarantee the confinement of the three-body system even at the ends of the segment $\varphi \in (0, \pi/3)$ where the potential $\Omega^{(3)}(\varrho, \varphi)$ itself vanishes.

The proof proceeds as follows. Firstly, assuming that ϱ is sufficiently large we show that with respect to the angle φ , there exist two minima in the *complete* SC potential

$$\Omega^{(SC)}(\varrho, \varphi) = \alpha^2 \varrho^3 \sin 3\varphi + \frac{\beta^2}{\varrho^2 \sin^2 3\varphi} \quad \alpha^2 = \frac{3}{2}\gamma > 0 \quad \beta^2 = \frac{9}{2}v(v+1) > 0. \tag{21}$$

They occur at the two ϱ -dependent angles $\varphi_{\pm} = \pi/6 - \varphi_{\mp}$. Their ϱ -dependence is controlled by the rule $\partial_{\varphi}\Omega^{(SC)} = 0$ with the following solution and its consequence:

$$\sin^2 3\varphi_{\pm} = \frac{2\beta^2}{\alpha^2 \varrho^5} \implies \Omega^{(SC)} > \frac{3\varrho}{2}(2\alpha^4 \beta^2 \varrho)^{1/3} \quad \varrho \geq \varrho_0 = \left(\frac{2\beta^2}{\alpha^2}\right)^{1/5}.$$

This means that the potential $\Omega^{(SC)}$ grows sufficiently quickly as a function of ϱ in the whole $\varrho \gg 1$ asymptotic wedge. The model possesses just the discrete spectrum.

Marginally we may note that the latter proof may be generalized from the above very special cubic model to all the potentials $\Omega^{(2M+1)}(\varrho, \varphi)$. Indeed, as long as the angular minimum moves quite quickly to the wedge boundary with the growing M , $\varphi_- = \pi/3 - \varphi_+ \sim \varrho^{-(2M+3)/3}$, the insertion of this value in any $M > 1$ potential produces the general estimate

$$\Omega = \alpha^2 \Omega^{(2M+1)} + \beta^2 \Omega^{(-2)} \sim \varrho^{4M/3} \quad \varrho \gg 1.$$

We see that for all the integer exponents $M > 0$, no particle can escape to infinity. The potential grows at least as a $4M/3$ th power of ϱ in the whole interval of the angles in the asymptotic domain of ϱ .

3.3. An ultimate illustration: $A = 3$ spectrum at $K = L = 3$

The climax of our analysis and considerations comes with the strong-repulsion construction of the energies in any power-law $A = 3$ system. For the sake of brevity we shall pick, *pars pro toto*, the same unsolvable spiked cubic oscillator as above, described by the non-separable PD Schrödinger equation

$$\left[-\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\varrho^2} \frac{\partial^2}{\partial \varphi^2} + \Omega^{(SC)}(\varrho, \varphi) \right] \psi(\varrho, \varphi) = E \psi(\varrho, \varphi) \quad \beta^2 \gg 1. \tag{22}$$

We already know that its potential has a double-well shape with two minima at every fixed and sufficiently large radial distance $\varrho \gg 1$ from the origin. With the decrease of ϱ , both these minima decrease and their positions move towards the central maximum (which ‘sits’ at $\varphi = \varphi_0 = \pi/6$). All these three extremes merge at $\varrho = \varrho_0 = (2\beta^2/\alpha^2)^{1/5}$. Below this distance, just a single minimum survives in the middle of the valley, at $\varphi = \varphi_0$.

The potential at its minimum still decreases with the further decrease of ϱ , and the unique absolute minimum of $\Omega^{(SC)}(\varrho, \varphi)$ is finally reached at $\varphi = \varphi_0 = \pi/6$ and at the distance

$$R = \varrho_0^{(\min)} = \sqrt[5]{2\beta^2/(3\alpha^2)}.$$

This quantity is large whenever $\beta^2 \gg 1$ so that we may use it as a new, slightly simpler measure of the strength of the SC two-body repulsion. This means that we may re-define $\beta^2 = 3\alpha^2 R^5/2$, $\varrho = R + \xi$, $\varphi = \pi/6 + \eta/R$ and expand

$$\begin{aligned} \frac{1}{\alpha^2} \Omega^{(SCO)}(\varrho, \varphi) &= \varrho^3 \sin 3\varphi + \frac{\beta^2}{\alpha^2 \varrho^2 \sin^2 3\varphi} \\ &= \frac{5}{2}R^3 + 9R\eta^2 + \frac{15}{2}R\xi^2 - \frac{81}{2}\xi\eta^2 - 5\xi^3 \\ &\quad + \frac{1}{R} \left(\frac{675}{8}\eta^4 + 27\xi^2\eta^2 + \frac{15}{2}\xi^4 \right) + \mathcal{O}\left(\frac{1}{R^2}\right). \end{aligned}$$

We see that the corrections decrease with the integer powers of $1/R$. This enables us to insert their series in equation (22). With a new re-scaling constant $\sigma = 1/\sqrt[4]{R}$ introduced in the two independent variables,

$$\varrho = R + \sigma x \quad \varphi = \pi/6 + \sigma y/R$$

this gives our final approximate Schrödinger equation

$$\left[-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + \alpha^2 \left(\frac{15}{2}x^2 + 9y^2 \right) + \mathcal{O}(\sigma^5) \right] \psi(x, y) = \sigma^2 \left(E - \frac{5}{2}R^3 \right) \psi(x, y). \quad (23)$$

The reliability of this approximation will increase with the growth of $R \gg 1$ or $\sigma = \sigma(R) \ll 1$.

At this point one must emphasize the *two* parallel merits of approximation (23). Firstly, it is a separable PD equation which may be analysed as two independent OD equations. Secondly, both these OD components are exactly solvable. This means that in the strongly spiked domain, the low-lying energies are given by the following closed formula:

$$E = E_{m,n} = \frac{5}{2}R^3 + \alpha\sqrt{\frac{15R}{2}}(2m+1) + 3\alpha\sqrt{R}(2n+1) + \mathcal{O}\left(\frac{1}{R^{3/4}}\right). \quad (24)$$

We see that these energies are numbered by the respective ‘radial’ and ‘angular’ quantum numbers $m, n = 0, 1, \dots$ in a way which resembles the previous Calogeroian exactly solvable case. Thus, one may expect that the same asymptotic construction remains applicable to all our models (A4).

4. Summary

Two main messages are conveyed by our present paper:

- (a) the power-law confinement of three spinless particles in one dimension leads to a comparatively simple PD Schrödinger equation, provided only that we work in the spherical Jacobi coordinates;
- (b) a quantum osculation method generates the low-lying spectra with a precision which increases with the strength of the repulsive two-body core in $V(x_i - x_j)$.

An exceptional, unexpected separability of the quartic anharmonic oscillations of three particles has also been revealed as a by-product of our systematic explicit construction of the corresponding PD Hamiltonian operators.

The former pair of messages resulted, basically, from a successful generalization of the Calogero trigonometric identity (B3). In the historical perspective of [2], the latter identity opened the path towards the explicit construction of the whole class of solvable models. Some of the consequences of its present generalizations are equally exciting: this is the essence of our mathematical achievement.

One of its most important uses for physics may be seen in the context of the large- β expansions. Their construction at any $K \neq 2 \neq L$ seems to be significantly facilitated by a generic survival of the separability *beyond* the Calogero exceptional model whenever we move to the *strongly repulsive regime*. In this sense, even all the multi-term generalizations of our present power-law $A = 3$ oscillators characterized, in both the long-range and short-range regime, by a polynomial behaviour of their two-body potentials remain tractable as models which will be *approximately separable and solvable* in the domain of the very strong two-body repulsion at short distances.

One of the most remarkable purely technical aspects of our present work appeared to be an *unexpectedly slow* growth of the complexity of the closed trigonometric formulae for the total potentials $\Omega^{(m)}(\varrho, \varphi)$ with increasing absolute values of their maximal-power superscripts m .

This is a lucky circumstance which significantly enhances the feasibility of the construction of the approximate spectra in polar coordinates and in the strongly-spiked limit.

In the future, one may expect that also a systematic incorporation of the higher-order corrections will remain feasible. We plan a more systematic study of such a possibility based on a refined strong-coupling perturbation-series expansion of the observable quantities in powers of the auxiliary quantity $1/R$ where R denotes the radial distance of the absolute minimum of $\Omega^{(m)}(\varrho, \varphi)$ and grows in proportion to the strength of the repulsive two-body core.

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Appendix A. Jacobi coordinates at $A = 3$

It is well known that the introduction of the suitably normalized centre-of-mass coordinate $Z = Z^{(A)} = \sum_{k=1}^A x_k / \sqrt{A}$ enables the elimination of the free bulk motion. Of course, the use of the new variable Z imposes a constraint upon the old A -plet of coordinates. The redundancy of one of the positions x_k may be easily resolved by the transition to the multiplet (Z, X, Y, \dots) of the so-called Jacobi coordinates which may be defined at any number of particles A and which we shall specify, at $A = 3$, by their most transparent matrix definition

$$\begin{pmatrix} Z \\ X \\ Y \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (\text{A1})$$

with an elementary inversion. Thus, the original particle coordinates may be understood as a triplet of linear functions of three new parameters,

$$\begin{pmatrix} x_1(Z, X, Y) \\ x_2(Z, X, Y) \\ x_3(Z, X, Y) \end{pmatrix} = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} Z \\ X \\ Y \end{pmatrix}. \quad (\text{A2})$$

These functions enter the general interaction term in equation (2),

$$\hat{W}(x_1, x_2, x_3) = \sum_{m=-K}^L F_m W^{(m)}(x_1, x_2, x_3) \quad F_L = \alpha^2 > 0 \quad F_{-K} = \beta^2 > 0 \quad (\text{A3})$$

$$W^{(m)}(x_1, x_2, x_3) = (x_1 - x_2)^m + (x_2 - x_3)^m + (x_3 - x_1)^m$$

where the positivity of F_L guarantees the confinement while the positivity of F_{-K} prevents the system from a collapse. In such notation we may re-write our $A = 3$ Schrödinger equation as a reduced PD problem in the two Cartesian coordinates X and Y ,

$$\left\{ -\frac{\partial^2}{\partial X^2} - \frac{\partial^2}{\partial Y^2} + \sum_{m=-K}^L F_m U^{(m)}(X, Y) - E \right\} \Phi(X, Y) = 0 \quad (\text{A4})$$

where the functions

$$U^{(m)}(X, Y) \equiv W^{(m)}[x_1(Z, X, Y), x_2(Z, X, Y), x_3(Z, X, Y)]$$

prove independent of Z . One may also employ the polar re-parametrization of the Jacobi coordinates,

$$X = X(\varrho, \varphi) = \varrho \sin \varphi \quad Y = Y(\varrho, \varphi) = \varrho \cos \varphi \quad (\text{A5})$$

arriving at an alternative, equivalent formulation of the same Schrödinger equation,

$$\left\{ -\frac{\partial^2}{\partial \varrho^2} - \frac{1}{\varrho^2} \frac{\partial^2}{\partial \varphi^2} + \sum_{m=-K}^L F_m \Omega^{(m)}(\varrho, \varphi) - E \right\} \Phi[X(\varrho, \varphi), Y(\varrho, \varphi)] = 0 \tag{A6}$$

where we introduced another abbreviation

$$\Omega^{(m)}(\varrho, \varphi) \equiv U^{(m)}[X(\varrho, \varphi), Y(\varrho, \varphi)].$$

Appendix B. Auxiliary trigonometric identities

Any practical analysis of the Schrödinger $A = 3$ equation (2) in Jacobi coordinates requires the explicit specification of its interaction term (3). Separately, let us derive its form for the three specific subclasses of the underlying two-body input force.

B.1. Potentials $U^{(m)}$ and $\Omega^{(m)}$ with even $m = 2M$

In our present notation, the regular two-body interactions are characterized by the even integer superscripts and by the following asymptotically confining interaction functions:

$$U^{(2M)}(X, Y) = [x_1(R, X, Y) - x_2(R, X, Y)]^{2M} + [x_2(R, X, Y) - x_3(R, X, Y)]^{2M} + [x_3(R, X, Y) - x_1(R, X, Y)]^{2M}.$$

The growth of M means strengthening of the confinement in both infinities, which has a definite phenomenological appeal. The explicit form of the functions $U^{(2M)}(X, Y)$ is, unfortunately, more and more complicated for the larger and larger M ,

$$U^{(6)}(X, Y) = \frac{33}{4}X^6 + \frac{45}{4}X^4Y^2 + \frac{135}{4}X^2Y^4 + \frac{27}{4}Y^6$$

$$U^{(8)}(X, Y) = \frac{3}{8}(X^2 + Y^2)(27Y^6 + 225X^2Y^4 - 15X^4Y^2 + 43X^6)$$

etc. Still, our search for the parallels between the solvable and unsolvable models finds its reward here since with the polar coordinates where we introduce an abbreviation

$$\Omega^{(2M)}(\varrho, \varphi) = U^{(2M)}[X(\varrho, \varphi), Y(\varrho, \varphi)] = \{x_1[R, X(\varrho, \varphi), Y(\varrho, \varphi)] - x_2[R, X(\varrho, \varphi), Y(\varrho, \varphi)]\}^{2M} + \dots$$

fairly perceivable simplifications emerge after patient trigonometric manipulations. This is one of our most important technical results. We get the sequence of pleasantly compact formulae,

$$\Omega^{(6)}(\varrho, \varphi) = \frac{3\varrho^6}{4}(9 + 2 \sin^2 3\varphi) = \frac{3\varrho^6}{4}(10 - \cos 6\varphi)$$

$$\Omega^{(8)}(\varrho, \varphi) = \frac{3\varrho^8}{8}(27 + 16 \sin^2 3\varphi)$$

$$\Omega^{(10)}(\varrho, \varphi) = \frac{27\varrho^{10}}{16}(9 + 10 \sin^2 3\varphi)$$

$$\Omega^{(12)}(\varrho, \varphi) = \frac{729\varrho^{12}}{32} + \frac{81\varrho^{12}}{2} \sin^2 3\varphi + \frac{3\varrho^{12}}{4} \sin^4 3\varphi$$

etc. A remarkable regularity emerges in this pattern. For example, the choice of the superscripts $m = 2M = 6N, 6N + 2$ or $6N + 4$ leads to the compact general formula

$$\frac{\Omega^{(2M)}(\varrho, \varphi)}{\varrho^{2M}} = \frac{3^M}{2^{M-1}} + c_1 \sin^2 3\varphi + \dots + c_N \sin^{2N} 3\varphi \tag{B1}$$

with a ϱ -independence on the right-hand side and with an extremely elementary N - and M -dependence of some of its coefficients which may be derived very easily whenever needed.

B.2. Potentials $U^{(m)}$ and $\Omega^{(m)}$ with odd $m = 2M + 1$

Systematic study of all the potentials $U^{(m)}$ with the *odd* powers $m = 3, 5, \dots$ is not so easily extended to $A = 4$, but in the present $A = 3$ setting it is well defined and worth the study. Elementary experiment quickly teach us that up to the trivial $U^{(1)}(X, Y) = 0$ we have to deal with the interesting functions. One cannot be surprised by the observation that the complexity of the representation of the complete potentials in Jacobi coordinates grows quite quickly with their superscript,

$$\begin{aligned} U^{(3)}(X, Y) &= \frac{3}{2}\sqrt{2}X(X^2 - 3Y^2) \\ U^{(5)}(X, Y) &= \frac{15}{4}\sqrt{2}(X^2 - 3Y^2)(X^2 + Y^2)X \\ U^{(7)}(X, Y) &= \frac{63}{8}\sqrt{2}(X^2 - 3Y^2)X(X^2 + Y^2)^2 \\ U^{(9)}(X, Y) &= \frac{3}{16}\sqrt{2}(81Y^6 + 279X^2Y^4 + 219X^4Y^2 + 85X^6)(X^2 - 3Y^2)X \end{aligned}$$

etc. Fortunately, in the same manner as above, the use of polar coordinates makes the geometric structure and symmetries of these two-dimensional potential wells much more evident,

$$\begin{aligned} \Omega^{(3)}(\varrho, \varphi) &= \frac{3\varrho^3}{2}\sqrt{2}\sin 3\varphi \\ \Omega^{(5)}(\varrho, \varphi) &= \frac{15\varrho^5}{4}\sqrt{2}\sin 3\varphi \\ \Omega^{(7)}(\varrho, \varphi) &= \frac{63\varrho^7}{8}\sqrt{2}\sin 3\varphi \\ \Omega^{(9)}(\varrho, \varphi) &= \frac{3\varrho^9}{16}\sqrt{2}\sin 3\varphi(81 + 4\sin^2 3\varphi) \left[= \frac{3\varrho^9}{16}\sqrt{2}\sin 3\varphi(83 - 2\cos 6\varphi) \right] \\ \Omega^{(11)}(\varrho, \varphi) &= \frac{33\varrho^{11}}{32}\sqrt{2}\sin 3\varphi(27 + 4\sin^2 3\varphi) \\ \Omega^{(13)}(\varrho, \varphi) &= \frac{117\varrho^{13}}{64}\sqrt{2}\sin 3\varphi(27 + 8\sin^2 3\varphi) \end{aligned}$$

etc. Again, once we have $m = 6N + 3$ or $m = 6N + 5$ or $m = 6N + 7$, the explicit form of the potential $\Omega^{(m)}$ will be prescribed by the trigonometric formula

$$\frac{\Omega^{(2M+1)}(\varrho, \varphi)}{\varrho^{2M+1}} = (2M + 1)\sqrt{2}\sin 3\varphi \left(\frac{3^{M-1}}{2^M} + d_1 \sin^2 3\varphi + \dots + d_N \sin^{2N} 3\varphi \right) \quad (\text{B2})$$

where, formally,

$$N = \text{entier} \left[\frac{M-1}{3} \right] \quad M = 1, 2, \dots$$

and $\text{entier}[x]$ denotes the integer part of a real number x .

B.3. Strongly singular potentials $\Omega^{(m)}$ with negative m

Using the same abbreviations as above we may immediately complement the Calogero identity (8) by its Coulombic predecessor,

$$\Omega^{(-1)}(\varrho, \varphi) = -\frac{3}{\varrho\sqrt{2}\sin 3\varphi} \quad \Omega^{(-2)}(\varrho, \varphi) = \left(\frac{3}{\varrho\sqrt{2}\sin 3\varphi}\right)^2 = [\Omega^{(-1)}(\varrho, \varphi)]^2. \quad (\text{B3})$$

These two formulae exhaust the set of standard singular forces. Nevertheless, there is no physical reason for avoiding the more singular repulsion near the origin, and the simplicity of the latter two formulae attracts attention to the more general spikes in

$$\Omega^{(-k)}(\varrho, \varphi) = (x_1 - x_2)^{-k} + (x_2 - x_3)^{-k} + (x_3 - x_1)^{-k}. \quad (\text{B4})$$

At the first three ‘non-trivial’ exponents $k > 2$ we reveal again the same overall structure of the interaction terms in their trigonometric representation,

$$\begin{aligned} \Omega^{(-3)}(\varrho, \varphi) &= [\Omega^{(-1)}(\varrho, \varphi)]^3 \left(1 - \frac{4}{9} \sin^2 3\varphi\right) \\ \Omega^{(-4)}(\varrho, \varphi) &= [\Omega^{(-1)}(\varrho, \varphi)]^4 \left(1 - \frac{16}{27} \sin^2 3\varphi\right) \\ \Omega^{(-5)}(\varrho, \varphi) &= [\Omega^{(-1)}(\varrho, \varphi)]^5 \left(1 - \frac{20}{27} \sin^2 3\varphi\right). \end{aligned}$$

Obviously, the pattern remains precisely the same as above. In the place of any routine classifications, let us only add the first less trivial formula

$$\Omega^{(-6)}(\varrho, \varphi) = [\Omega^{(-1)}(\varrho, \varphi)]^6 \left(1 - \frac{904}{667} \sin^2 3\varphi + \frac{8248}{18\,009} \sin^4 3\varphi\right)$$

which nicely illustrates not only the emergence of the new terms but also the close parallel between the positive and negative superscripts.

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